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DEGREE OF CONVERGENCE OF DERIVED FOURIER SERIES IN BESOV SPACES

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Abstract. In this paper, we study the degree of convergence of the functions of derived Fourier series in Besov spaces using Matrix-generalized Nörlund ($AN^{p,q}$) means. We also study an application of our main results.

Keywords: Degree of convergence, modulus of smoothness, Besov spaces, matrix-generalized Nörlund ($AN^{p,q}$) method, derived Fourier series.

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1 Introduction

The Besov space $B_\rho^\beta(L_q)$ is a set of functions f from L_q which have smoothness β and the parameter ρ gives a finer gradation of smoothness (see 5). It is a tool to describe the smoothenss properties of functions and contains a large number of fundamental spaces such as Sobolev, Lipschitz and Hölder spaces. These spaces appear naturally in many fields of analysis. Currently, there are two definitions of Besov spaces which are in use. First one uses Fourier transforms and the second uses modulus of smootheness of the function f. These two definitions are equivalent only with certain restrictions on the parameter; e.g. they are different when $q < 1$ and β is small.

The Besov spaces defined by the modulus of smoothness appear more naturally in many areas of analysis including approximation theory (Devore & Popov, 1988).

In this paper, we study the degree of convergence of the functions of derived Fourier series in Besov norms using a new summability matrix-generalized Nörlund ($AN^{p,q}$) means. However, detailed objectives of this paper will be presented in Section-3.

Organization of the paper is as follows: In Section-2, we give important definitions related to our work. In Section-3, we mention detailed objectives of the proposed problems and obtain the results. Application and the numerical result is discussed in Section-4 while conclusion is given in Section-5.

2 Notation and Preliminaries

In this section, we present some important notations and definitions.

2.1 Notation

$$\omega_\nu(f,y)_q = \begin{cases} \omega_1(f,y)_q, & 0<\beta<1 \\ \omega_2(f,y)_q & 1\le\beta<2. \end{cases}$$

2.2 Besov Spaces

For $1 \le q < \infty$, the space $L_q[0,2\pi]$ consists of all measurable functions on $[0,2\pi]$ such that

$$\int_0^{2\pi} \left|f(y)\right|^q dy < \infty$$

and the norm is defined by

$$||f||_q = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f(y)\right|^q dy\right)^{\frac{1}{q}}, & 1\le q<\infty; \\ ess \sup\limits_{f\in(0,2\pi)} |f(y)|, & q=\infty. \end{cases}$$

When $q = 2$ then,

$$||f||_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(y)|^2 dy\right)^{\frac{1}{2}}.$$

The ν-th order modulus of smoothness of a function $f \in L_q$ is defined by

$$\omega_\nu(f,y)_q = \sup_{0<h\le y} ||\Delta_h^\nu(f,\cdot)||_q,\ y>0 \tag{1}$$

where

$$\Delta_h^\nu(f,y) = \sum_{j=0}^{\nu}(-1)^{\nu-j}\binom{\nu}{j} f(y+jh), \quad \nu\in\mathbb{N}.$$

For $\nu = 1$ and $q = \infty$, $\omega_1(f,y)$ is called the modulus of continuity of f (Devore & Lorentz, 1993).
If $f \in C_{2\pi}$ and $\omega(f,y) = \mathcal{O}(y^\beta)$, for $0<\beta\le 1$, then the function $f \in Lip(\beta)$. If the function f belongs to $L_q, 0<q<\infty$, and $\omega(f,y)_q = \mathcal{O}(y^\beta), 0<\beta\le 1$, then the function $f \in Lip(\beta,q)$.
If $q=\infty$, then the class $Lip(\beta,q)$ reduces to the class $Lip(\beta)$. Thus,

$$Lip(\alpha) \subseteq Lip(\beta,q). \tag{2}$$

Let $\beta > 0$ and for $\nu > \beta$ suppose that $\nu = [\beta]+1$, where ν is the smallest ineteger. For $f \in L_q$, if

$$\omega_\nu(f,y)_q = \mathcal{O}(y^\beta) \tag{3}$$

then the function f belongs to the generalized Lipschitz class $Lip^*(\beta,q)$, $y>0$ and the seminorm of this class is given by

$$|f|_{Lip^*(\beta,q)} = \sup_{y>0}(y^{-\beta}\omega_\nu(f,y)_q). \tag{4}$$

Let $\beta > 0$ be given, and let $\nu = [\beta] + 1$. For $0 < q, \rho \leq \infty$, the Besov space $B^{\beta}_{\rho}(L_q)$ is the collection of all the signals (2π-periodic functions) $f \in L_q$ such that

$$|f|_{B^{\beta}_{\rho}(L_q)} = ||\omega_{\nu}(f,.)||_{\beta,\rho} = \begin{cases} \left[\int_0^{\pi}\left(y^{-\beta}\omega_{\nu}(f,y)_q\right)^{\rho}\frac{dy}{y}\right]^{\frac{1}{\rho}}, & 1 \leq \rho < \infty; \\ \sup\limits_{y>0}\left(y^{-\beta}\omega_{\nu}(f,y)_q\right), & \rho = \infty. \end{cases} \tag{5}$$

is finite (Prössdorf, 1975). It is known that (5) is a seminorm if $1 \leq q, \rho \leq \infty$ and a quasi-seminorm in other cases (Devore & Lorentz, 1993). Thus, the quasi-norm for $B^{\beta}_{\rho}(L_q)$ is defined by

$$||f||_{B^{\beta}_{\rho}(L_q)} = ||f||_q + |f|_{B^{\beta}_{\rho}(L_q)} = ||f||_q + ||\omega_{\nu}(f,.)||_{\beta,\rho}. \tag{6}$$

When $q = 2$, the quasi-norm for $B^{\beta}_{\rho}(L_2)$ is defined by

$$||f||_{B^{\beta}_{\rho}(L_2)} = ||f||_2 + |f|_{B^{\beta}_{\rho}(L_2)} = ||f||_2 + ||\omega_{\nu}(f,.)||_{\beta,\rho}. \tag{7}$$

Remark 1. *(i) In particular, for $\rho = \infty$, $B^{\beta}_{\infty}(L_q) = Lip^*(\beta, q)$.*

(ii) When $0 < \beta < 1$, the space $B^{\beta}_{\infty}(L_q)$ reduces to the generalized Hölder spaces $H_{\beta,q}$ (Das et al., 1996).

(iii) By taking $q = \infty = \rho$ and $0 < \beta < 1$, the Besov spaces reduces to the Hölder spaces H_{β} (Prössdorf, 1975).

2.3 Derived Fourier Series

Let f be a 2π-periodic Lebesgue integrable function defined on $[-\pi, \pi]$.
The Fourier series of f is given by

$$f(t) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty}(a_{\nu}\cos\nu t + b_{\nu}\sin\nu t). \tag{8}$$

The derived Fourier series of (8) is given by

$$f^{'}(t) \sim \sum_{\nu=1}^{\infty}(b_{\nu}\cos\nu t - a_{\nu}\sin\nu t) \tag{9}$$

which is obtained by differentiating (8) term by term.
The ν^{th} partial sum of (9) is given by

$$s^{'}_{\nu}(f^{'};t) = s^{'}_{\nu}(t) - f^{'}(t) = \frac{1}{2\pi}\int_0^{\pi} D_{\nu}(s)dg_t(s), \tag{10}$$

where

$$g_t(s) = f(t+s) - f(t-s) - 2sf^{'}(t)$$

and

$$dg_t(s) = d(f(t+s) - f(t-s)) - 2f^{'}(t)ds.$$

2.4 Matrix-Generalized Nörlund $(AN^{p,q})$ Means

Now we introduce for the first time a new product summability Matrix-Generalized Nörlund $(AN^{p,q})$ method.

Let $A = (a_{\nu,k});\ \ \nu, k = 0, 1, 2, \cdots$ be an infinite triangular matrix satisfying the Silverman-Toeplitz (Toeplitz, 1913) conditions of regularity i.e.

$$\begin{aligned}\sum_{k=0}^{\nu} a_{\nu,k} &= 1 \text{ as } \nu \to \infty,\\ a_{\nu,k} &= 0, \text{ for } k > \nu,\\ \sum_{k=0}^{\nu} |a_{\nu,k}| &\le M, \text{ a finite constant.}\end{aligned} \tag{11}$$

Let $\sum_{\nu=0}^{\infty} u_\nu$ be an infinite series such that $s_\nu = \sum_{k=0}^{\nu} u_k$. If $t_\nu^A = \sum_{k=0}^{\nu} a_{\nu,k} s_k \to s$ as $\nu \to \infty$, then the series $\sum_{\nu=0}^{\infty} u_\nu$ or sequence $\{s_\nu\}$ is summable to s by matrix (A) method.

Let $\{p_\nu\}$ and $\{q_\nu\}$ be the sequence of constants, real or complex such that

$$P_\nu = p_0 + p_1 + \cdots + p_\nu = \sum_{k=0}^{\nu} p_k \to \infty, \text{as } \nu \to \infty,$$

$$Q_\nu = q_0 + q_1 + \cdots + q_\nu = \sum_{k=0}^{\nu} q_k \to \infty, \text{as } \nu \to \infty,$$

and

$$R_\nu = p_0 q_\nu + p_1 q_{\nu-1} + \cdots + p_\nu q_0 = \sum_{k=0}^{\nu} p_k q_{\nu-k} \to \infty, \text{as } \nu \to \infty.$$

Given two sequences $\{p_\nu\}$ and $\{q_\nu\}$, convolution $(p * q)$ is defined as

$$R_\nu = (p * q)_\nu = \sum_{j=0}^{\nu} p_{\nu-j} q_j.$$

We write

$$t_\nu^{N^{p,q}} = \frac{1}{R_\nu} \sum_{j=0}^{\nu} p_{\nu-j} q_j s_j$$

If $R_\nu \neq 0$ for all ν, generalized Nörlund transform $(N^{p,q})$ of the sequence $\{s_\nu\}$ is the sequence $\{t_\nu^{N^{p,q}}\}$.

If $\{t_\nu^{N^{p,q}}\} \to s$, as $\nu \to \infty$, then the series $\sum_{\nu=0}^{\infty} u_\nu$ or sequence $\{s_\nu\}$ is summable to s by generalized Nörlund $(N^{p,q})$ method and is denoted by $s_\nu \to s(N^{p,q})$.

The necessary and sufficient condition for $(N^{p,q})$ method to be regular are

$$\sum_{j=0}^{\nu} |p_{\nu-j} q_j| = \mathcal{O}(|R_\nu|) \text{ and } p_{\nu-j} = o(|R_\nu|) \text{ as } \nu \to \infty$$

for every fixed $j \ge 0$ for which $q_j \neq 0$.

If the matrix method is superimposed on the generalized Nörlund $(N^{p,q})$ method, then a new summability $AN^{p,q}$ method is obtained. We can define $AN^{p,q}$ product method as

$$\begin{aligned} t_\nu^{AN^{p,q}} &= \sum_{j=0}^{\nu} a_{\nu,j} t_j^{N^{p,q}} \\ &= \sum_{j=0}^{\nu} a_{\nu,j} \frac{1}{R_j} \sum_{k=0}^{j} p_{j-k} q_k s_k. \end{aligned}$$

If $t_\nu^{AN^{p,q}} \to s$ as $\nu \to \infty$, then the series $\sum_{\nu=0}^{\infty} u_\nu$ or the sequence $\{s_\nu\}$ is summable to s by $AN^{p,q}$ method.

The regularity condition of $AN^{p,q}$ method is as follows:

$$\begin{aligned} s_\nu \to s \Rightarrow t_\nu^{AN^{p,q}} &\to s, \text{as } \nu \to \infty \text{ so } N^{p,q} \text{ method is regular}, \\ &\Rightarrow A(t_\nu^{N^{p,q}}) = t_\nu^{AN^{p,q}} \to s \text{ as } \nu \to \infty \text{ so } A \text{ method is regular}, \\ &\Rightarrow (AN^{p,q}) \text{ method is regular.} \end{aligned}$$

Remark 2. *Consider an infinite series*

$$1 + \sum_{\nu=1}^{\infty} (-1)^\nu \cdot 2\nu. \tag{12}$$

The ν^{th} partial sum of the series is given by

$$s_\nu = \begin{cases} \nu + 1, & \nu \text{ is even,} \\ 0, & \nu \text{ is odd.} \end{cases}$$

If we take $a_{\nu,k} = \frac{1}{\nu+1}$ for $k \le \nu$, then series (12) is not summable by matrix means.
If we take $p_\nu = 1$ and $q_\nu = 1$ for all $\nu \ge 0$ in the generalized Nörlund means, then series (12) is also not summable by generalized Nörlund means.

$$\begin{aligned} t_\nu^{AN^{p,q}} &= \sum_{j=0}^{\nu} a_{\nu,j} \frac{1}{R_j} \sum_{k=0}^{j} p_{j-k} q_k s_k \\ &= \sum_{j=0}^{\nu} a_{\nu,j} \frac{1}{j+1} [p_j q_0 s_0 + p_{j-1} q_1 s_1 + \cdots p_0 q_j s_j] \\ &= \sum_{j=0}^{\nu} a_{\nu,j} \frac{1}{j+1} [s_0 + s_1 + \cdots + s_j] \\ &= a_{\nu,0}[s_0] + a_{\nu,1}\left[\frac{s_0 + s_1}{2}\right] + \cdots + a_{\nu,\nu}\left[\frac{s_0 + s_1 + \cdots + s_\nu}{\nu + 1}\right]. \end{aligned}$$

Clearly, we have seen that series (12) is summable by matrix-generalized Nörlund ($AN^{p,q}$) means. Therefore, the product means is more powerful than the single means.

Remark 3. *Particular cases of $AN^{p,q}$ means:*

(i) *$AN^{p,q}$ means reduces to $C^1 N^{p,q}$ when $a_{\nu,k} = \frac{1}{(\nu+1)}$.*

(ii) *$AN^{p,q}$ means reduces to $HN^{p,q}$ when $a_{\nu,k} = \frac{1}{(\nu-k+1)\log \nu}$.*

(iii) *$AN^{p,q}$ means reduces to $C^\delta N^{p,q}$ when $a_{\nu,k} = \frac{\binom{\nu-k+\delta-1}{\delta-1}}{\binom{\nu+\delta}{\delta}}$.*

(iv) $AN^{p,q}$ *means reduces to* $H^pN^{p,q}$ *when* $a_{\nu,k} = \dfrac{1}{\log^{p-1}(\nu+1)\prod_{m=0}^{p-1}\log^m(k+1)}$.

(v) $AN^{p,q}$ *means reduces to* $N^pN^{p,q}$ *when* $a_{\nu,k} = \dfrac{p_{\nu-k}}{P_\nu}$, *where* $P_\nu = \sum_{k=0}^{\infty} p_k$.

(vi) $AN^{p,q}$ *means reduces to* $\bar{N}^pN^{p,q}$ *when* $a_{\nu,k} = \dfrac{p_k}{P_\nu}$, *where* $P_\nu = \sum_{k=0}^{\infty} p_k$.

(vii) $AN^{p,q}$ *means reduces to* AN^p *when* $q_\nu = 1$, $\forall\nu$.

(viii) $AN^{p,q}$ *means reduces to* $A\bar{N}^q$ *when* $p_\nu = 1$, $\forall\nu$.

(ix) $AN^{p,q}$ *means reduces to* AC^δ *when* $p_\nu = \binom{\nu+\delta-1}{\delta-1}$, $\delta > 0$ *and* $q_\nu = 1$, $\forall\nu$.

Remark 4. *The above particular cases can be further reduced as:*

(i) $C^1N^{p,q}$ *means reduces to* C^1N^p *when* $q_\nu = 1$, $\forall\nu$.

(ii) $HN^{p,q}$ *means reduces to* HN^p *when* $q_\nu = 1$, $\forall\nu$.

(iii) $C^\delta N^{p,q}$ *means reduces to* $C^\delta N^p$ *when* $q_\nu = 1$, $\forall\nu$.

(iv) $H^pN^{p,q}$ *means reduces to* H^pN^p *when* $q_\nu = 1$, $\forall\nu$.

(v) $\bar{N}^qN^{p,q}$ *means reduces to* $\bar{N}^qN^p$ *when* $p_\nu = 1$, $\forall\nu$.

(vi) $C^1N^{p,q}$ *means reduces to* $C^1\bar{N}^q$ *when* $p_\nu = 1$, $\forall\nu$.

(vii) $C^\delta N^{p,q}$ *means reduces to* $C^\delta\bar{N}^q$ *when* $p_\nu = 1$, $\forall\nu$.

(viii) $HN^{p,q}$ *means reduces to* $H\bar{N}^q$ *when* $p_\nu = 1$, $\forall\nu$.

(ix) $H^pN^{p,q}$ *means reduces to* $H^p\bar{N}^q$ *when* $p_\nu = 1$, $\forall\nu$.

(x) $HN^{p,q}$ *means reduces to* HC^δ *when* $p = \binom{\nu+\delta-1}{\delta-1}$, $\delta > 0$ *and* $q_\nu = 1$, $\forall\nu$.

(xi) $H^pN^{p,q}$ *means reduces to* H^pC^δ *when* $p = \binom{\nu+\delta-1}{\delta-1}$, $\delta > 0$ *and* $q_\nu = 1$, $\forall\nu$.

(xii) $N^pN^{p,q}$ *means reduces to* N^pC^δ *when* $p = \binom{\nu+\delta-1}{\delta-1}$, $\delta > 0$ *and* $q_\nu = 1$, $\forall\nu$.

(xiii) $\bar{N}^qN^{p,q}$ *means reduces to* $\bar{N}^qC^\delta$ *when* $p = \binom{\nu+\delta-1}{\delta-1}$, $\delta > 0$ *and* $q_\nu = 1$, $\forall\nu$.

2.5 Degree of Convergence

The degree of convergence of a summation method to a given function f is a measure that how fast T_ν converges to f, which is given by

$$||f - T_\nu|| = \mathcal{O}\left(\frac{1}{\lambda_\nu}\right) \text{ (London, 2008)},$$

where $\lambda_\nu \to \infty$ as $\nu \to \infty$.

3 Main Result

In this section, we present our main result to find the degree of convergence of derived Fourier series in Besov spaces using Matrix-generalized Nörlund ($AN^{p,q}$) means.

3.1 Degree of Convergence of a Function of Derived Fourier Series

The degree of approximation of a function in function spaces viz, Lipschtiz, Hölder and generalized Hölder class using different means of Fourier series, has been studied by the authors Rhoades (2014); Nigam & Hadish (2018); Krasniqi & Szal (2019); Nigam & Rani (2020) etc.

In this subsection, we study the degree of convergence of a function in Besov spaces using a new summability method matrix-generalized Nörlund ($AN^{p,q}$) means of derived Fourier series and establish the following theorems. We observe that the results obtained in the following theorems provide best approximation of function $f^{'}$ in Besov norms.

Remark 5. *Since the derived Fourier series converges uniformly in L_2-norm, we will find the degree of convergence of derived Fourier series in L_2-norm.*

Theorem 1. *Let $f^{'}$ be a 2π-period and Lebesgue integrable function belonging to Besov spaces $B^{\beta}_{\rho}(L_2)$, $1 < \rho < \infty$, then for $0 \leq \gamma < \beta < 2$, the degree of convergence of a function $f^{'}$ of derived Fourier series using $AN^{p,q}$ transform, is given by*

$$\begin{aligned}||T_\nu(\cdot)||_{B^{\gamma}_{\rho}(L_2)} =&\mathcal{O}\bigg((\nu+1)\int_0^{\frac{1}{\nu+1}}|dg_t(s)| + \frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^{\pi}\frac{|dg_t(s)|}{s^2}\bigg)\\ &+\mathcal{O}\bigg((\nu+1)\int_0^{\frac{1}{\nu+1}}\big(s^{-\gamma-\frac{1}{\rho}}\big)|dg_t(s)|\\ &+\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^{\pi}\big(s^{-\gamma-\frac{1}{\rho}-2}\big)|dg_t(s)|\bigg).\end{aligned}$$

Following lemmas are required for the proof of Theorem 1.

Lemma 1. *If $\{p_\nu\}$ and $\{q_\nu\}$ are monotonic increasing and monotonic decreasing sequences respectively, then*

$$(\nu+1)p_\nu q_0 = \mathcal{O}(R_\nu). \tag{13}$$

Proof. Since

$$\begin{aligned}R_\nu = \sum_{k=0}^{\nu} p_k q_{\nu-k} &= p_0q_\nu + p_1q_{\nu-1} + \cdots + p_\nu q_0\\ &\geq p_0q_\nu + p_0q_\nu + \cdots + p_0q_\nu\\ &= (\nu+1)p_0q_\nu.\end{aligned}$$

Thus,

$$(\nu+1)p_\nu q_0 = \mathcal{O}(R_\nu).$$

□

Lemma 2. *For $0 < s \leq \frac{1}{\nu+1}$, $M_\nu^{AN^{p,q}}(s) = \mathcal{O}(\nu+1)$.*

Proof. For $0 < s \leq \frac{1}{\nu+1}$, $\quad \sin(\frac{s}{2}) \geq \frac{s}{\pi}$ and $\sin(k+\frac{1}{2})s \leq (k+\frac{1}{2})s$.

$$\begin{aligned}
|M_\nu^{AN^{p,q}}(s)| =& \left|\frac{1}{2\pi}\sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k \frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\right| \\
\leq& \frac{1}{4\pi}\left|\sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k \frac{(2k+1)s}{\frac{s}{\pi}}\right| \\
\leq& \frac{1}{4}\left|\sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k(2k+1)\right| \\
\leq& \frac{1}{4}\left|\sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}(2j+1)\sum_{k=0}^{j} p_{j-k}q_k\right| \\
=& \frac{1}{4}\left|\sum_{j=0}^{\nu} a_{\nu j}(2j+1)\right|.
\end{aligned}$$

Thus,

$$M_\nu^{AN^{p,q}}(s) = \mathcal{O}(\nu+1).$$

□

Lemma 3. *For* $\frac{1}{\nu+1} < s \leq \pi$, $M_\nu^{AN^{p,q}}(s) = \mathcal{O}\left(\frac{1}{s^2(\nu+1)}\right)$.

Proof. For $\frac{1}{\nu+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $|\sin s| \leq 1$.

$$\begin{aligned}
|M_\nu^{AN^{p,q}}(s)| =& \left|\frac{1}{2\pi}\sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k \frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\right| \\
\leq& \frac{1}{2s}\left|\sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k \sin\left(k+\frac{1}{2}\right)s\right|.
\end{aligned}$$

Now, using Abel's transformation we have,

$$\begin{aligned}
\left|\sum_{k=0}^{j} p_{j-k}q_k \sin\left(k+\frac{1}{2}\right)s\right| =& \left|\sum_{k=0}^{j-1}(p_{j-k}q_k - p_{j-k-1}q_{k+1})\sum_{r=0}^{k}\sin\left(r+\frac{1}{2}\right)s \right. \\
& \left. + p_0 q_j \sum_{k=0}^{j}\sin\left(k+\frac{1}{2}\right)s\right| \\
=& \mathcal{O}\left(\frac{1}{s}\right)\left[\sum_{k=0}^{j-1}\left|p_{j-k}q_k - p_{j-k-1}q_{k+1}\right| + |p_0 q_j|\right] \\
=& \mathcal{O}\left(\frac{p_j q_0}{s}\right).
\end{aligned}$$

Thus, by using Lemma 1, we have

$$M_\nu^{AN^{p,q}}(s) = \mathcal{O}\left(\frac{1}{s^2(\nu+1)}\right).$$

□

Proof of Theorem 1. Using (10), the $AN^{p,q}$ transform of the sequence $\{s_\nu^{'}(t)\}$ is given by

$$T_\nu^{'}(t) = t_\nu^{AN^{p,q}}(t) - f^{'}(t) = \sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k\{s_k^{'}(t) - f^{'}(t)\}$$

$$= \sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k\left[\frac{1}{2\pi}\int_0^\pi \frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}dg_t(s)\right] \tag{14}$$

Thus,

$$T_\nu^{'}(t) = \frac{1}{2\pi}\int_0^\pi \sum_{j=0}^{\nu} a_{\nu j}\frac{1}{R_j}\sum_{k=0}^{j} p_{j-k}q_k\frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}dg_t(s) \tag{15}$$

$$= \int_0^\pi M_\nu^{AN^{p,q}}(s)dg_t(s). \tag{16}$$

By definition of Besov norm given in (7), we have

$$||T_\nu(\cdot)||_{B_\rho^\gamma(L_2)} = ||T_\nu(\cdot)||_2 + ||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho}. \tag{17}$$

Using generalized Minkowski's inequality (Chui, 1992), we have

$$||T_\nu(\cdot)||_2 \leq \int_0^\pi |dg_t(s)||M_\nu(s)|$$

$$= \int_0^{\frac{1}{\nu+1}} |dg_t(s)||M_\nu(s)| + \int_{\frac{1}{\nu+1}}^{\pi} |dg_t(s)||M_\nu(s)|. \tag{18}$$

Now, using Lemmas 2 and 3, we have

$$||T_\nu(\cdot)||_2 = \mathcal{O}\bigg((\nu+1)\int_0^{\frac{1}{\nu+1}} |dg_t(s)| + \frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^{\pi} \frac{|dg_t(s)|}{s^2}\bigg). \tag{19}$$

Now, using definition of Besov spaces, we have

$$||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho} \leq \left\{\int_0^\pi \Big(y^{-\gamma}||T_\nu(\cdot,y)||_2\Big)^\rho \frac{dy}{y}\right\}^{\frac{1}{\rho}}. \tag{20}$$

Using generalized Minkowski's inequality (Chui (1992)), we have

$$||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho} \leq \bigg[\int_0^\pi \bigg(\int_0^\pi |M_\nu(s)||dg_t(s)|\bigg)^\rho \frac{dy}{y^{\gamma\rho+1}}\bigg]^{\frac{1}{\rho}}$$

$$= \int_0^\pi |M_\nu(s)||dg_t(s)|\bigg(\int_0^\pi \frac{dy}{y^{\gamma\rho+1}}\bigg)^{\frac{1}{\rho}}$$

$$= \int_0^\pi |M_\nu(s)||dg_t(s)|\bigg(\int_0^s \frac{dy}{y^{\gamma\rho+1}}\bigg)^{\frac{1}{\rho}} \tag{21}$$

$$+ \int_0^\pi |M_\nu(s)||dg_t(s)|\bigg(\int_s^\pi \frac{dy}{y^{\gamma\rho+1}}\bigg)^{\frac{1}{\rho}}.$$

Using the second mean value theorem, we have

$$||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho} \leq \int_0^\pi |M_\nu(s)||dg_t(s)|\big(s^{-\gamma-\frac{1}{\rho}}\big)$$

$$= \int_0^\pi \big(s^{-\gamma-\frac{1}{\rho}}\big)|M_\nu(s)||dg_t(s)|$$

$$= \int_0^{\frac{1}{\nu+1}} \big(s^{-\gamma-\frac{1}{\rho}}\big)|M_\nu(s)||dg_t(s)| + \int_{\frac{1}{\nu+1}}^{\pi} \big(s^{-\gamma-\frac{1}{\rho}}\big)|M_\nu(s)||dg_t(s)|. \tag{22}$$

Using Lemmas 2 and 3, we have

$$\begin{aligned}||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho} =&\mathcal{O}\Bigg((\nu+1)\int_0^{\frac{1}{\nu+1}}\big(s^{-\gamma-\frac{1}{\rho}}\big)|dg_t(s)| \\ &+\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^{\pi}\big(s^{-\gamma-\frac{1}{\rho}-2}\big)|dg_t(s)|\Bigg). \end{aligned} \tag{23}$$

Combining (17), (19) and (23), we have

$$\begin{aligned}||T_\nu(\cdot)||_{B^\gamma_\rho(L_2)} =&\mathcal{O}\Bigg((\nu+1)\int_0^{\frac{1}{\nu+1}}|dg_t(s)|+\frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^{\pi}\frac{|dg_t(s)|}{s^2}\Bigg) \\ &+\mathcal{O}\Bigg((\nu+1)\int_0^{\frac{1}{\nu+1}}\big(s^{-\gamma-\frac{1}{\rho}}\big)|dg_t(s)| \\ &+\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^{\pi}\big(s^{-\gamma-\frac{1}{\rho}-2}\big)|dg_t(s)|\Bigg).\end{aligned}$$

□

Remark 6. *When $\rho=\infty$, the Besov space $B^\beta_\infty(L_q)$, $\beta\geq 0$, $q\geq 1$ reduces to generalized Lipschitz class $Lip^*(\beta,q)$ and the corresponding norm $||\cdot||_{B^\beta_\infty(L_q)}$ is given by*

$$||f||_{B^\beta_\infty(L_q)}=||f||_{Lip^*(\beta,q)}=||f||_q+\sup_{y>0}y^{-\beta}\omega_\nu(f,y)_q. \tag{24}$$

Thus, in view of Remark 6, we establish the following theorem to obtain degree of convergence for $f^{'}\in Lip^*(\beta,2)$, $q=2$, $\rho=\infty$:

Theorem 2. *Let $f^{'}$ be a 2π-period and Lebesgue integrable function belonging to generalized Lipschitz spaces $Lip^*(\beta,L_2)$, $\rho=\infty$, then for $0\leq\gamma<\beta<2$, the degree of convergence of a function $f^{'}$ of derived Fourier series using $AN^{p,q}$ transform, is given by*

$$\begin{aligned}||T_\nu(\cdot)||_{Lip^*(\gamma,L_2)} =&\mathcal{O}\Bigg((\nu+1)\int_0^{\frac{1}{\nu+1}}|dg_t(s)|ds+\frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^{\pi}\frac{|dg_t(s)|}{s^2}\Bigg) \\ &+\mathcal{O}(\nu+1)\sup_{0<y,s\leq\pi}\Bigg(\int_0^{\frac{1}{\nu+1}}y^{-\gamma}|dg_t(s)|\Bigg) \\ &+\mathcal{O}\Bigg(\frac{1}{\nu+1}\Bigg)\sup_{0<y,s\leq\pi}\Bigg(\int_{\frac{1}{\nu+1}}^{\pi}y^{-\gamma-2}|dg_t(s)|\Bigg).\end{aligned}$$

Proof of Theorem 2. By definition of Besov norm given in (24), we have

$$||T_\nu(\cdot)||_{Lip^*(\gamma,L_2)}=||T_\nu(\cdot)||_2+||\omega_k(T_\nu,\cdot)_2||_{\gamma,\infty}. \tag{25}$$

Using the generalized Minkowski's inequality (Chui, 1992), we have

$$\begin{aligned}
||\omega_k(T_{\nu,\cdot})_2||_{\gamma,\infty} &= \sup_{0<y,s\leq\pi}(y^{-\gamma}\omega_k(T_\nu,y)_2)\\
&= \sup_{0<y,s\leq\pi}(y^{-\gamma}||T_\nu(\cdot,y)||_2)\\
&\leq \sup_{0<y,s\leq\pi}\left[y^{-\gamma}\left(\int_0^\pi |M_\nu(s)||dg_t(s)|\right)\right]\\
&= \sup_{0<y,s\leq\pi}\left(\int_0^{\frac{1}{\nu+1}} y^{-\gamma}|M_\nu(s)||dg_t(s)| + \int_{\frac{1}{\nu+1}}^\pi y^{-\gamma}|M_\nu(s)||dg_t(s)|\right)\\
&=\mathcal{O}(\nu+1)\sup_{0<y,s\leq\pi}\left(\int_0^{\frac{1}{\nu+1}} y^{-\gamma}|dg_t(s)|\right)\\
&\quad+\mathcal{O}\left(\frac{1}{\nu+1}\right)\sup_{0<y,s\leq\pi}\left(\int_{\frac{1}{\nu+1}}^\pi y^{-\gamma-2}|dg_t(s)|\right). \qquad (26)
\end{aligned}$$

From equation (19) and (26), we have

$$\begin{aligned}
||T_\nu(\cdot)||_{Lip^*(\gamma,L_2)} =&\mathcal{O}\left((\nu+1)\int_0^{\frac{1}{\nu+1}}|dg_t(s)|ds + \frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^\pi \frac{|dg_t(s)|}{s^2}\right)\\
&+\mathcal{O}(\nu+1)\sup_{0<y,s\leq\pi}\left(\int_0^{\frac{1}{\nu+1}} y^{-\gamma}|dg_t(s)|\right)\\
&+\mathcal{O}\left(\frac{1}{\nu+1}\right)\sup_{0<y,s\leq\pi}\left(\int_{\frac{1}{\nu+1}}^\pi y^{-\gamma-2}|dg_t(s)|\right).
\end{aligned}$$

□

Corollary 1. *Let f' be a 2π-period and Lebesgue integrable function belonging to Besov spaces $B^\beta_\rho(L_2)$, $1<\rho<\infty$, then for $0\leq\gamma<\beta<2$, the degree of convergence of a function f' of derived Fourier series using $C^1N^{p,q}$ transform, is given by*

$$\begin{aligned}
||T_\nu(\cdot)||_{B^\gamma_\rho(L_2)} =&\mathcal{O}\left((\nu+1)\int_0^{\frac{1}{\nu+1}}|dg_t(s)| + \frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^\pi \frac{|dg_t(s)|}{s^2}\right)\\
&+\mathcal{O}\left((\nu+1)\int_0^{\frac{1}{\nu+1}}(s^{-\gamma-\frac{1}{\rho}})|dg_t(s)|\right.\\
&\left.+\frac{1}{(\nu+1)}\int_{\frac{1}{\nu+1}}^\pi (s^{-\gamma-\frac{1}{\rho}-2})|dg_t(s)|\right).
\end{aligned}$$

Remark 7. *We can deduce further corollaries from the Theorem 1 in view of Remarks 3 and 4.*

Corollary 2. *Let f' be a 2π-period and Lebesgue integrable function belonging to generalized Lipschitz spaces $Lip^*(\beta,L_2)$, $\rho=\infty$, then for $0\leq\gamma<\beta<2$, the degree of convergence of a function f' of derived Fourier series using $AN^{p,q}$ transform, is given by*

$$\begin{aligned}
||T_\nu(\cdot)||_{Lip^*(\gamma,L_2)} =&\mathcal{O}\left((\nu+1)\int_0^{\frac{1}{\nu+1}}|dg_t(s)|ds + \frac{1}{\nu+1}\int_{\frac{1}{\nu+1}}^\pi \frac{|dg_t(s)|}{s^2}\right)\\
&+\mathcal{O}(\nu+1)\sup_{0<y,s\leq\pi}\left(\int_0^{\frac{1}{\nu+1}} y^{-\gamma}|dg_t(s)|\right)\\
&+\mathcal{O}\left(\frac{1}{\nu+1}\right)\sup_{0<y,s\leq\pi}\left(\int_{\frac{1}{\nu+1}}^\pi y^{-\gamma-2}|dg_t(s)|\right).
\end{aligned}$$

Remark 8. *We can deduce further corollaries from the Theorem 2 in view of Remarks 3 and 4.*

4 Application

In this section, we study an application of our main results.

4.1 Application on the degree of convergence of a function of derived Fourier series in Besov norm using Matrix-generalized Nörlund $(AN^{p,q})$ means

Consider a function $f^{'}(t) = \sin t$ and $a_{\nu,k} = \frac{1}{(\nu-k+1)\log\nu}$ for $\nu \leq k$, and $a_{\nu,k} = 0$ for $\nu > k$, and $p_\nu = \binom{\nu+\delta-1}{\delta-1}$, $\delta > 0$, $q_\nu = 1$, $\forall\nu$.

Thus $dg_t(s) = -2\cos t(\sin^2 \frac{s}{2})ds$.

Therefore, $M_\nu^{AN^{p,q}} = \mathcal{O}(\nu+1)$ for $0 < s \leq \frac{1}{\nu+1}$ and $M_\nu^{AN^{p,q}} = \mathcal{O}\bigg(\frac{1}{s^2(\nu+1)}\bigg)$ for $\frac{1}{\nu+1} < s \leq \pi$.

Taking $\beta = 1$, $\gamma = 0$ and $\rho = \infty$.

Since $|\cos s| \leq 1$ and $\sin(\frac{s}{2}) \geq (\frac{s}{\pi})$, for $0 < s \leq \pi$, therefore we have,

$$||T_\nu(\cdot)||_2 = \mathcal{O}\bigg(\frac{\pi}{(\nu+1)}\bigg)$$

and

$$||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho} = \mathcal{O}\bigg(\frac{1}{(\nu+1)} + \frac{\log(\pi(\nu+1))}{(\nu+1)}\bigg).$$

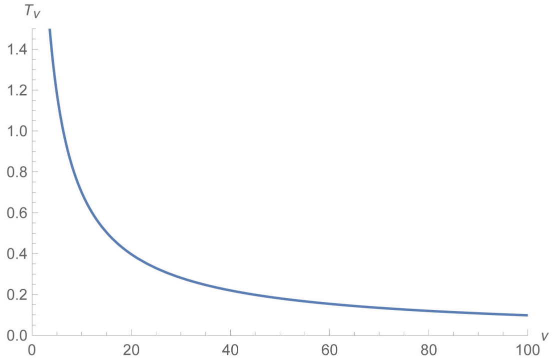
Thus, the degree of convergence of $f^{'}(t) = \sin t$ is obtained by

$$\begin{aligned}||T_\nu^{'}(\cdot)||_{B_\beta^\gamma(L_2)} &= ||T_\nu(\cdot)||_2 + ||\omega_k(T_\nu,\cdot)_2||_{\gamma,\rho}\\ &= \mathcal{O}\bigg(\frac{\pi+1+\log(\pi(\nu+1))}{(\nu+1)}\bigg).\end{aligned}$$

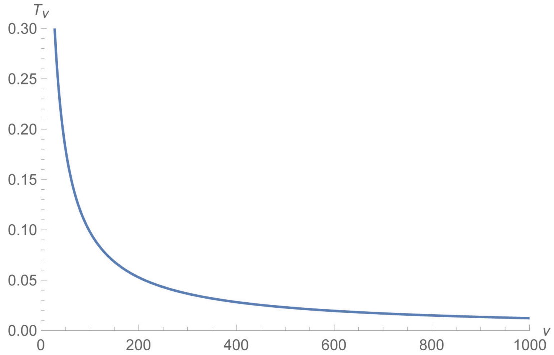
Table 1

ν	$T_\nu^{'}(t) = \frac{\pi+1+\log(\pi(\nu+1))}{(\nu+1)}$
100	0.0980341
1000	0.0121829
10000	0.0014495
50000	0.0003221
75000	0.0002202
100000	0.0001679
.	.
.	.
.	.
∞	0

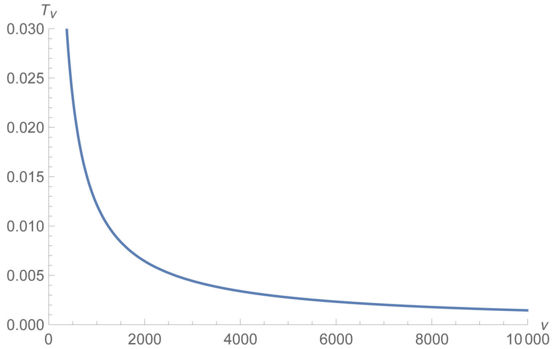
Now, we draw the following graphs of $T_\nu^{'}(f^{'})$ for different values of ν:



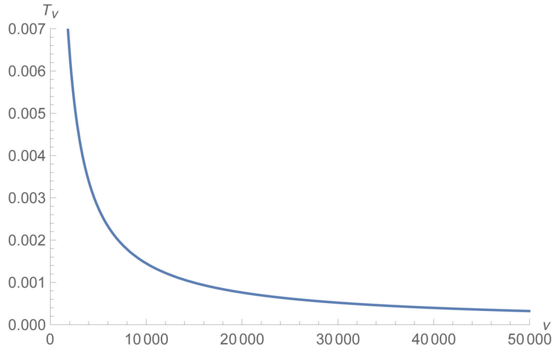
1(a): For $\nu = 100$



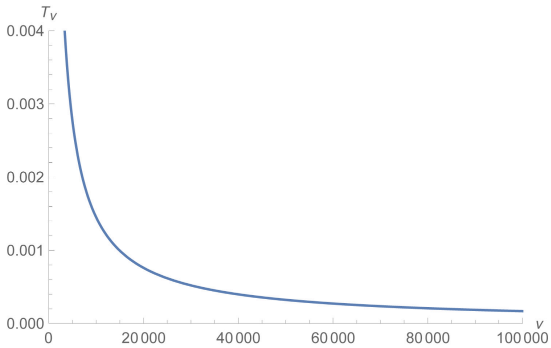
1(b): For $\nu = 1000$



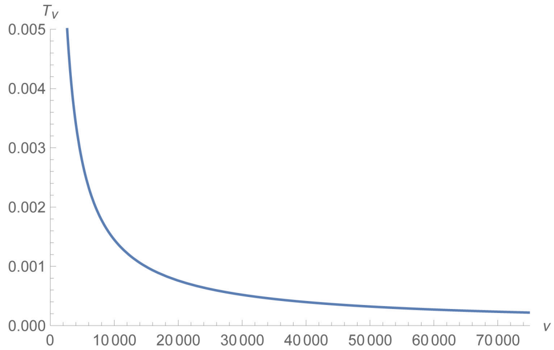
1(c): For $\nu = 10000$



1(d): For $\nu = 50000$



1(e): For $\nu = 75000$



1(f): For $\nu = 100000$.

Figure 1: Degree of convergence of $f^{'}(t) = \sin(t)$.

Remark 9. *From the Table* 1 *and figures* $1(a)$ *to* $1(f)$*, we observe that the results obtained in Theorem 1 and 2 together for* $1 < \rho \leq \infty$ *provide best approximation of the function* $f^{'}$.

5 Conclusion

From the Table 1 and figures $1(a)$ to $1(f)$, we observe that the results obtained in Theorem 1 and 2 together for $1 < \rho \leq \infty$ provide best approximation of the function $f^{'}$.

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